

# Algebro-geometric solution of the coupled Burgers equation

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## Abstract

We derive theta function representation of algebro-geometric solution of a Coupled Burgers equation which the second nonlinear evolution equation in a hierarchy. We also derive the algebro-geometric characters of the meromorphic function  $\phi$  and the Baker-Akhiezer vector  $\Psi$ .

**keywords:** algebro-geometric solution; algebraic curves; Riemann theta functions

## 1 Introduction

According to inverse spectral theory and algebro-geometric methods, we can construct the explicit theta function representations of quasi-periodic solutions of integrable nonlinear evolution equations(including, soliton solutions as special limiting cases or quasi-periodic behavior of nonlinear phenomenon) [1, 2, 3, 4], and this approach developed by pioneers such as Novikov, Dubrovin, McKean, Lax, Cao [5, 6, 7, 8, 9]. How to obtain explicit solutions of the soliton equations, to reveal inherent structure of soliton equations and to describe the characteristic for the integrability of soliton equations, there are also many works had been done [10, 11, 12].

In this paper, we will focus on a derivation of a coupled system and search for the algebro-geometric solution of the following on the basis of approaches in [3, 13, 14]:

$$\begin{aligned} u_t &= v_{xx} - 2(uv)_x, \\ v_t &= u_{xx} + 2uu_x - 6vv_x, \end{aligned} \tag{1.1}$$

obviously, when  $u = v$ , this system is the Burgers equation

$$u_t + 4uu_x - u_{xx} = 0,$$

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so we call the above system(1.1) as the coupled Burgers equation.

The original motivation of this paper was to construct algebro-geometric solution of the coupled Burgers equation based on its obtained Lax pairs. The paper is organized as follows. In Sect. 2, we describe our zero-curvature formalism and derive the coupled Burgers equation. In Sect. 3, we establish a direct relation between the elliptic variables and the potentials. Our principal Sect. 4 is devoted to a detailed derivation of theta function formulas of all algebro-geometric quantities involved.

## 2 The hierarchy and Lax pairs of the coupled Burgers equation

In this section, we introduce the Lenard gradient sequence  $\{G_j\}_{j=-1,0,1,\dots}$  to derive the hierarchy associated with equations (1.1) by the recursion relation

$$KG_{j-1} = JG_j, \quad j = 0, 1, 2, \dots \quad G_j|_{(u,v)=0} = 0, \quad G_{-1} = (1, 1)^T, \quad (2.1)$$

where  $G_j = (G_j^{(1)}, G_j^{(2)})$  and two skew-symmetric operators  $(\partial = \partial/\partial x)$

$$K = \begin{pmatrix} 0 & \partial^2 - 2\partial u \\ -\partial^2 - 2u\partial & -2\partial v - 2v\partial \end{pmatrix}, \quad J = \begin{pmatrix} 2\partial & 0 \\ 0 & -2\partial \end{pmatrix}. \quad (2.2)$$

A direct calculation gives from the recursion relation(2.1)that

$$G_0 = (-u, v)^T, \quad G_1 = (\frac{1}{2}v_x - uv, -\frac{1}{2}u_x - \frac{1}{2}u^2 + \frac{3}{2}v^2)^T. \quad (2.3)$$

Consider the spectral problem

$$\psi_x = U\psi, \quad U = \begin{pmatrix} -\lambda + v & u + v \\ u - v & \lambda - v \end{pmatrix} \quad (2.4)$$

and the auxiliary problem:

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix}, \quad (2.5)$$

where

$$V_{11}^{(m)} = -\sum_{j=0}^m 2(-\lambda + v)G_{j-1}^{(2)}\lambda^{m-j} - \sum_{j=0}^m G_{j-1,x}^{(1)}\lambda^{m-j},$$

$$V_{12}^{(m)} = \sum_{j=0}^m [(G_{j-1,x}^{(2)} - G_{j-1,x}^{(1)}) - 2(u + v)G_{j-1}^{(2)}]\lambda^{m-j},$$

$$V_{21}^{(m)} = \sum_{j=0}^m [(G_{j-1,x}^{(2)} + G_{j-1,x}^{(1)} - 2(u-v)G_{j-1}^{(2)})\lambda^{m-j}].$$

Then the compatibility condition of (2.4) and (2.5) is  $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$ , which is equivalent to the hierarchy of nonlinear evolution equations

$$\begin{aligned} u_{t_m} &= G_{m-1,xx}^{(2)} - 2(uG_{m-1}^{(2)})_x = 2G_{mx}^{(1)}, \\ v_{t_m} &= -G_{m-1,xx}^{(1)} - 2uG_{m-1,x}^{(1)} - 2(vG_{m-1}^{(2)})_x - 2vG_{m-1,x}^{(2)}, \end{aligned}$$

in brief,

$$(u_{t_m}, v_{t_m})^T = X_m, \quad m \geq 0, \quad (2.6)$$

and  $X_j = KG_{j-1} = JG_j$ . The first two nontrivial equations are

$$\begin{aligned} u_t &= -2u_x, \\ v_t &= -2v_x, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} u_t &= v_{xx} - 2(uv)_x, \\ v_t &= u_{xx} + 2uu_x - 6vv_x, \end{aligned} \quad (2.8)$$

the second system (2.8) is our called the coupled Burgers equation.

Assume that (2.4) and (2.5) have two basic solutions  $\psi = (\psi_1, \psi_2)^T$  and  $\phi = (\phi_1, \phi_2)^T$ . We define a matrix  $W$  by

$$W = \frac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma = \begin{pmatrix} G & F \\ H & -G \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.9)$$

By (2.4) and (2.5), we can find that

$$W_x = [U, W], \quad W_{t_m} = [V^{(m)}, W]. \quad (2.10)$$

Which implies that  $\partial_x \det W = 0$ ,  $\partial_{t_m} \det W = 0$ , and (2.10) can be written as

$$\begin{aligned} G_x &= (u+v)H - (u-v)F, \\ F_x &= 2F(-\lambda+v) - 2(u+v)G, \\ H_x &= 2(u-v)G - 2(-\lambda+v)H, \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} G_{t_m} &= HV_{12}^{(m)} - FV_{21}^{(m)}, \\ F_{t_m} &= 2FV_{11}^{(m)} - 2GV_{12}^{(m)}, \\ H_{t_m} &= 2GV_{21}^{(m)} - 2HV_{11}^{(m)}. \end{aligned} \quad (2.12)$$

Select  $a, b$  as the form

$$a = \sum_{j=0}^N a_{j-1} \lambda^{N-j}, \quad b = \sum_{j=0}^N b_{j-1} \lambda^{N-j},$$

and suppose the functions  $G, F, H$  have the following finite-order polynomials in  $\lambda$

$$\begin{aligned} G &= a_x + 2vb - 2\lambda b, \\ F &= a_x - b_x + 2(u+v)b, \\ H &= -a_x - b_x + 2(u-v)b. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.11), we have

$$K\overline{G} = \lambda J\overline{G}, \quad \overline{G} = (a, b)^T,$$

which is equivalent to

$$K\overline{G}_{j-1} = J\overline{G}_j, \quad J\overline{G}_{-1} = 0, K\overline{G}_{N-1} = 0, \quad \overline{G}_j = (a_j, b_j)^T. \tag{2.14}$$

It is easy to see that the equation  $J\overline{G}_{-1} = 0$  has the general solution  $\overline{G}_{-1} = \alpha_0 G_{-1} + \beta_0 \widehat{G}_{-1}$ ,  $\widehat{G}_{-1} = (1, 0)^T \in \text{Ker} J$ . Where  $\alpha_0$  and  $\beta_0$  are constants of integration, and we can obtain from (2.1) and (2.14) that

$$\overline{G}_k = \sum_{j=0}^{k+1} \alpha_j G_{k-j} + \beta_{k+1} \widehat{G}_{-1}, \quad -1 \leq k \leq N-1, \tag{2.15}$$

where  $\alpha_0, \alpha_1, \dots, \alpha_{k+1}$  and  $\beta_{k+1}$  are constants of integration. Substituting (2.15) into (2.14) yields the stationary equation

$$\sum_{j=0}^N \alpha_j X_{N-j} = \alpha_0 X_N + \alpha_1 X_{N-1} + \dots + \alpha_N X_0 = 0, \tag{2.16}$$

this means that expression (2.11) are existent. Without loss of generality, Let  $\alpha_0 = 1, \beta_0 = 0$ , from (2.14) and (2.15), we have

$$\begin{aligned} \overline{G}_{-1} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \overline{G}_0 = \begin{pmatrix} u + \alpha_1 + \beta_1 \\ v + \alpha_1 \end{pmatrix}, \\ \overline{G}_1 &= \begin{pmatrix} \frac{1}{2}v_x - uv - \alpha_1 u + \alpha_2 + \beta_2 \\ -\frac{1}{2}u_x - \frac{1}{2}u^2 + \frac{3}{2}v^2 + \alpha_1 v + \alpha_2 \end{pmatrix}. \end{aligned}$$

### 3 Evolution of elliptic variables

Now we suppose that the functions  $G, F, H$  are finite-order polynomials in  $\lambda$  from (2.13)

$$G = \sum_{j=0}^{N+1} g_j \lambda^{N+1-j}, \quad F = \sum_{j=0}^N f_j \lambda^{N-j}, \quad H = \sum_{j=0}^N h_j \lambda^{N-j}, \tag{3.1}$$

with

$$\begin{aligned}
g_0 &= -2b_{-1}, \\
g_j &= a_{j-2} + 2vb_{j-2} - 2b_{j-1}, \quad 1 \leq j \leq N, \\
g_{N+1} &= a_{N-1,x} + 2vb_{N-1}, \\
f_j &= a_{j-1,x} - b_{j-1,x} + 2(u+v)b_{j-1}, \quad 1 \leq j \leq N, \\
h_j &= -a_{j-1,x} - b_{j-1,x} + 2(u-v)b_{j-1}, \quad 1 \leq j \leq N.
\end{aligned} \tag{3.2}$$

Therefore it is easy to calculate the first few members

$$\begin{aligned}
g_0 &= -2, \quad g_1 = -2\alpha_1, \quad g_2 = 2u_x + u^2 + v^2 - 2\alpha_2, \\
f_0 &= 2(u+v), \quad f_1 = u_x - v_x + 2v(u+v) + 2\alpha_1(u+v), \\
f_2 &= \frac{1}{2}u_{xx} + \frac{1}{2}v_{xx} - 2u_xv - uv_x - \alpha_1u_x - \alpha_1v_x - 3vv_x \\
&\quad - u^3 + 3uv^2 + 2\alpha_1uv + 2\alpha_2u - u^2v + 3v^3 + 2\alpha_1v^2 + 2\alpha_2v, \\
h_0 &= 2(u-v), \quad h_1 = -u_x - v_x + 2v(u-v) + 2\alpha_1(u-v), \\
h_2 &= \frac{1}{2}u_{xx} - \frac{1}{2}v_{xx} + 2u_xv + uv_x + \alpha_1u_x - \alpha_1v_x - 3vv_x \\
&\quad - u^3 + 3uv^2 + 2\alpha_1uv + 2\alpha_2u + u^2v - 3v^3 - 2\alpha_1v^2 - 2\alpha_2v.
\end{aligned} \tag{3.3}$$

We can write  $F$  and  $H$  as polynomials of  $\lambda$  to define the elliptic coordinates  $u_i$  and  $v_i$

$$F = 2(u+v) \prod_{i=1}^N (\lambda - u_i), \quad H = 2(u-v) \prod_{i=1}^N (\lambda - v_i). \tag{3.4}$$

By comparing the coefficients of  $\lambda^{n-1}$  and  $\lambda^{n-2}$ , we get

$$f_1 = -2(u+v) \sum_{i=1}^N u_i, \quad h_1 = -2(u-v) \sum_{i=1}^N v_i, \tag{3.5}$$

$$f_2 = 2(u+v) \sum_{i < j} u_i u_j, \quad h_2 = 2(u-v) \sum_{i < j} v_i v_j. \tag{3.6}$$

Thus from (3.3) and (3.5), we have

$$\begin{aligned}
-\frac{1}{2} \frac{(u-v)_x}{u+v} - v - \alpha_1 &= \sum_{j=1}^N u_j, \\
\frac{1}{2} \frac{(u+v)_x}{u-v} - v - \alpha_1 &= \sum_{j=1}^N v_j,
\end{aligned} \tag{3.7}$$

from (3.3) and (3.6), we have

$$\begin{aligned}
\frac{1}{4} \frac{(u+v)_{xx}}{u+v} - \frac{\alpha_1}{2} \frac{(u+v)_x}{u+v} + \frac{3v^2}{2} + \alpha_1v + \alpha_2 - \frac{u^2}{2} - v \frac{(u+v)_x}{u+v} - \frac{v_x}{2} &= \sum_{i < j} u_i u_j, \\
\frac{1}{4} \frac{(u-v)_{xx}}{u-v} + \frac{\alpha_1}{2} \frac{(u-v)_x}{u-v} + \frac{3v^2}{2} + \alpha_1v + \alpha_2 - \frac{u^2}{2} - v \frac{(u-v)_x}{u-v} - \frac{v_x}{2} &= \sum_{i < j} v_i v_j.
\end{aligned} \tag{3.8}$$

Consider the function  $\det W$  which is a  $(2N + 2)$  th-order polynomial in  $\lambda$  with constant coefficients of the  $x$ - flow and  $t_m$ -flow

$$-\det W = G^2 + FH = 4 \prod_{j=1}^{2N+2} (\lambda - \lambda_j) = R(\lambda). \quad (3.9)$$

Substituting (3.1) into (3.9) and comparing the coefficients of  $\lambda^{2N+1}$  and  $\lambda^{2N}$  yields

$$\begin{aligned} 2g_0g_1 &= -4 \sum_{j=1}^{2N+2} \lambda_j, \\ 2g_0g_2 + g_1^2 + f_0h_0 &= 4 \sum_{i<j} \lambda_i\lambda_j, \end{aligned} \quad (3.10)$$

together with (3.3) gives

$$\alpha_1 = -\frac{1}{2} \sum_{j=1}^{2N+2} \lambda_j, \quad \alpha_2 = -\frac{1}{8} \left( \sum_{j=1}^{2N+2} \lambda_j \right)^2 + \frac{1}{2} \sum_{i<j} \lambda_i\lambda_j. \quad (3.11)$$

From (3.9), we get that

$$G|_{\lambda=u_k} = \sqrt{R(u_k)}, \quad G|_{\lambda=v_k} = \sqrt{R(v_k)}. \quad (3.12)$$

Noticing (2.11) and (3.4), we obtain

$$F_x|_{\lambda=u_k} = -2(u+v)u_{k,x} \prod_{j=1, j \neq k}^N (u_k - u_j) = -2(u+v)G|_{\lambda=u_k}, \quad (3.13)$$

$$H_x|_{\lambda=v_k} = -2(u-v)v_{k,x} \prod_{j=1, j \neq k}^N (v_k - v_j) = -2(u-v)G|_{\lambda=v_k}. \quad (3.14)$$

Hence we have the evolution of the elliptic coordinates along the  $x$  flows

$$u_{k,x} = \frac{\sqrt{R(u_k)}}{\prod_{j=1, j \neq k}^N (u_k - u_j)}, \quad 1 \leq k \leq N, \quad (3.15)$$

$$v_{k,x} = -\frac{\sqrt{R(v_k)}}{\prod_{j=1, j \neq k}^N (v_k - v_j)}, \quad 1 \leq k \leq N. \quad (3.16)$$

In a way similar to the above expression, by using (2.12) and (3.4) we arrive at

$$u_{k,t_m} = \frac{\sqrt{R(u_k)}(u+v)^{-1}V_{12}^{(m)}|_{\lambda=u_k}}{\prod_{i=1, i \neq k}^N (u_k - u_i)}, \quad 1 \leq k \leq N, \quad (3.17)$$

$$v_{k,t_m} = -\frac{\sqrt{R(v_k)}(u-v)^{-1}V_{21}^{(m)}|_{\lambda=v_k}}{\prod_{i=1, i \neq k}^N (v_k - v_i)}, \quad 1 \leq k \leq N, \quad (3.18)$$

and from (2.1), (2.2) and (2.5), we have

$$V_{12}^{(1)}|_{\lambda=u_k} = -2(u+v)u_k + (-u_x + v_x) - 2(u+v)v, \quad (3.19)$$

$$V_{21}^{(1)}|_{\lambda=v_k} = -2(u-v)v_k + (v_x + u_x) - 2(u-v)v, \quad (3.20)$$

$$\begin{aligned} V_{12}^{(2)}|_{\lambda=u_k} = & -2(u+v)u_k^2 + (u_x + v_x - 2uv - 2v^2)u_k - \frac{1}{2}u_{xx} + 3vv_x \\ & - \frac{1}{2}v_{xx} + 2u_xv + uv_x + u^3 - 3uv^2 + u^2v - 3v^3, \end{aligned} \quad (3.21)$$

$$\begin{aligned} V_{21}^{(2)}|_{\lambda=v_k} = & -2(u-v)v_k^2 + (v_x - u_x - 2uv + 2v^2)v_k - \frac{1}{2}u_{xx} + 3vv_x \\ & + \frac{1}{2}v_{xx} - 2u_xv - uv_x + u^3 - 3uv^2 - u^2v + 3v^3. \end{aligned} \quad (3.22)$$

## 4 Algebro-geometric solutions

In the following, we will give the algebro-geometric solution of the coupled Burgers Equation(1.1). Returning to (3.9), we naturally introduce the hyperelliptic curve  $\mathcal{K}_N$  of arithmetic genus  $N$  defined by

$$\mathcal{K}_N : y^2 - R(\lambda) = 0. \quad (4.1)$$

The curve  $\mathcal{K}_N$  can be compactified by joining two points at infinity  $P_{\infty\pm}$ ,  $P_{\infty+} = (P_{\infty-})^*$ . Still denoting its projective closure by  $\mathcal{K}_N$ . Here we assume that the zeros  $\lambda_j (j = 1, \dots, 2N+2)$  of  $R(\lambda)$  in (3.9) are mutually distinct, then the hyperelliptic curve  $\mathcal{K}_N$  becomes nonsingular. According to the definition of  $\mathcal{K}_N$ , we can lift the roots  $\{u_j\}_{j=1,\dots,N}$ ,  $\{v_j\}_{j=1,\dots,N}$  to  $\mathcal{K}_N$  by introducing

$$\hat{u}_j(x, t_m) = (u_j(x, t_m), G(u_j(x, t_m))), \quad (4.2)$$

$$\hat{v}_j(x, t_m) = (v_j(x, t_m), -G(v_j(x, t_m))), \quad (4.3)$$

where  $j = 1, \dots, N$ ,  $(x, t_m) \in \mathbb{R}^2$ . Moreover, from (4.1) we know that  $y^2 = G^2 + FH$ , that is  $(y+G)(y-G) = FH$ , then we can define the meromorphic function  $\phi(\cdot, x, t_m)$  on  $\mathcal{K}_N$

$$\phi(\cdot, x, t_m) = \frac{y+G}{F} = \frac{H}{y-G}, \quad (4.4)$$

where  $P = (\lambda, y) \in \mathcal{K}_N \setminus \{P_{\infty\pm}\}$ . Hence the divisor of  $\phi(P, x, t_m)$  is

$$(\phi(P, x, t_m)) = \mathcal{D}_{\hat{v}(x, t_m)P_{\infty+}}(P) - \mathcal{D}_{\hat{u}(x, t_m)P_{\infty-}}(P), \quad (4.5)$$

where

$$\hat{u}(x, t_m) = \{\hat{u}_1(x, t_m), \dots, \hat{u}_N(x, t_m)\} \in \text{Sym}^N(\mathcal{K}_N),$$

$$\hat{v}(x, t_m) = \{\hat{v}_1(x, t_m), \dots, \hat{v}_N(x, t_m)\} \in \text{Sym}^N(\mathcal{K}_N).$$

And the branch of  $y(\cdot)$  near  $P_{\infty\pm}$  is fixed according to

$$\lim_{\substack{|z| \rightarrow \infty \\ P \rightarrow P_{\infty\pm}}} \frac{y(P)}{G(z)} = \mp 1.$$

Based on the definition of meromorphic function  $\phi(\cdot, x, t_m)$  in (4.4), the spectral problem (2.4) and the auxiliary problem (2.5), we can define the Baker-Akhiezer vector  $\Psi(\cdot, x, x_0, t_m, t_{m,0})$  on  $\mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$  by

$$\Psi(\cdot, x, x_0, t_m, t_{m,0}) = \begin{pmatrix} \psi_1(\cdot, x, x_0, t_m, t_{m,0}) \\ \psi_2(\cdot, x, x_0, t_m, t_{m,0}) \end{pmatrix}, \quad (4.6)$$

where

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) = & \exp\left(\int_{x_0}^x ((-\lambda + v(x', t_m)) - (u(x', t_m) + v(x', t_m)))\phi(P, x', t_m)dx'\right. \\ & \left. + \int_{t_{m,0}}^{t_m} (V_{11}^{(m)}(\lambda, x_0, s) - V_{12}^{(m)}(\lambda, x_0, s))\phi(P, x_0, s)ds\right), \end{aligned} \quad (4.7)$$

$$\psi_2(P, x, x_0, t_m, t_{m,0}) = -\psi_1(P, x, x_0, t_m, t_{m,0})\phi(P, x, t_m), \quad (4.8)$$

with  $P \in \mathcal{K}_N, (x, t_m), (x_0, t_{m,0}) \in \mathbb{R}^2$ .

In the following, we introduce the Riemann surface  $\Gamma$  of the hyperelliptic curve  $\mathcal{K}_N$  and equip  $\Gamma$  with a canonical basis of cycles:  $a_1, a_2, \dots, a_N; b_1, b_2, \dots, b_N$  which are independent and have intersection numbers as follows

$$a_i \circ a_j = 0, \quad b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, i, j = 1, 2, \dots, N.$$

We will choose the following set as our basis

$$\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad l = 1, 2, \dots, N,$$

which are linearly independent homomorphic differentials from each other on  $\Gamma$ , and let

$$A_{ij} = \int_{a_j} \tilde{\omega}_i, \quad B_{ij} = \int_{b_j} \tilde{\omega}_i.$$

It is possible to show that the matrices  $A = (A_{ij})$  and  $B = (B_{ij})$  are  $N \times N$  invertible period matrices [15, 16]. Now we define the matrices  $C$  and  $\tau$  by  $C = (C_{ij}) = A^{-1}$ ,  $\tau = (\tau_{ij}) = A^{-1}B$ . Then the matrix  $\tau$  can be shown to symmetric ( $\tau_{ij} = \tau_{ji}$ ) and it has a positive-definite imaginary part ( $\text{Im } \tau > 0$ ). If we normalize  $\tilde{\omega}_j$  into the new basis  $\omega_j$

$$\omega_j = \sum_{l=1}^N C_{jl} \tilde{\omega}_l, \quad l = 1, 2, \dots, N,$$



then we have

$$\begin{aligned}\int_{a_j} \omega_j &= \sum_{l=1}^N C_{jl} \int_{a_j} \tilde{\omega}_l = \sum_{l=1}^N C_{jl} A_{li} = \delta_{ji}, \\ \int_{b_j} \omega_i &= \sum_{l=1}^N C_{jl} \int_{b_j} \tilde{\omega}_l = \sum_{l=1}^N C_{jl} B_{li} = \tau_{ji}.\end{aligned}$$

Now we define the Abel-Jacobi coordinates

$$\rho_j^{(1)}(x, t_m) = \sum_{k=1}^N \int_{P_0}^{\hat{u}_k(x, t_m)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N \int_{\lambda(P_0)}^{u_k} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad (4.9)$$

$$\rho_j^{(2)}(x, t) = \sum_{k=1}^N \int_{P_0}^{\hat{v}_k(x, t_m)} \omega_j = \sum_{k=1}^N \sum_{l=1}^N \int_{\lambda(P_0)}^{v_k} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad (4.10)$$

where  $\lambda(P_0)$  is the local coordinate of  $P_0$ . From (3.15) and (4.9), we get

$$\partial_x \rho_j^{(1)} = \sum_{k=1}^N \sum_{l=1}^N C_{jl} \frac{u_k^{l-1} u_{kx}}{\sqrt{R(u_k)}} = \sum_{k=1}^N \sum_{l=1}^N \frac{2C_{jl} u_k^{l-1}}{\prod_{j=1, j \neq k}^N (u_k - u_j)},$$

which implies

$$\partial_x \rho_j^{(1)} = 2C_{jN} = \Omega_j^{(1)}, \quad j = 1, 2, \dots, N, \quad (4.11)$$

With the help of the following equality

$$\sum_{k=1}^N \frac{u_k^{l-1}}{\prod_{i=1, i \neq k}^N (u_k - u_i)} = \begin{cases} \delta_{lN}, & l = 1, 2, \dots, N, \\ \sum_{i_1+i_2+\dots+i_N=l-N} u_1^{i_1} u_2^{i_2} \dots u_N^{i_N}, & l > N. \end{cases} \quad (4.12)$$

In a similar way, we obtain from (4.9), (4.10), (3.7), (3.15), (3.16), (3.17), (3.18), (3.19), (3.20):

$$\partial_t \rho_j^{(1)} = -2C_{j, N-1} + 2\alpha_1 C_{j, N} = \Omega_j^{(2)}, \quad j = 1, 2, \dots, N, \quad (4.13)$$

$$\partial_x \rho_j^{(2)} = -\Omega_j^{(1)}, \quad j = 1, 2, \dots, N, \quad (4.14)$$

$$\partial_t \rho_j^{(2)} = -\Omega_j^{(2)}, \quad j = 1, 2, \dots, N. \quad (4.15)$$

Let  $\mathcal{T}$  be the lattice generated by  $2n$  vectors  $\delta_j, \tau_j$ , where  $\delta_j = (\underbrace{0, \dots, 0}_{j-1}, 1, \underbrace{0, \dots, 0}_{n-j})$  and  $\tau_j = \tau \delta_j$ , the Jacobian variety of  $\Gamma$  is  $\mathcal{J} = \mathbb{C}^n / \mathcal{T}$ . On the basis of these results, we obtain the following

$$\rho_j^{(1)}(x, t) = \Omega_j^{(1)} x + \Omega_j^{(2)} t + \gamma_j^{(1)}, \quad (4.16)$$

$$\rho_j^{(2)}(x, t) = -\Omega_j^{(1)} x - \Omega_j^{(2)} t + \gamma_j^{(2)}, \quad (4.17)$$

where  $\gamma_j^{(i)}$  ( $i = 1, 2$ ) are constants, and

$$\begin{aligned}\underline{\rho}^{(1)} &= (\rho_1^{(1)}, \rho_2^{(1)}, \dots, \rho_N^{(1)})^T, \quad \underline{\rho}^{(2)} = (\rho_1^{(2)}, \rho_2^{(2)}, \dots, \rho_N^{(2)})^T, \\ \underline{\Omega}^{(m)} &= (\Omega_1^{(m)}, \Omega_2^{(m)}, \dots, \Omega_N^{(m)})^T, \quad \underline{\gamma}^{(m)} = (\gamma_1^{(m)}, \gamma_2^{(m)}, \dots, \gamma_N^{(m)})^T, \quad m = 1, 2.\end{aligned}$$

Now we introduce the Abel map  $\mathcal{A}(P) : \text{Div}(\Gamma) \rightarrow \mathcal{J}$

$$\begin{aligned}\mathcal{A}(P) &= \int_{P_0}^P \underline{\omega}, \quad \underline{\omega} = (\omega_1, \omega_2, \dots, \omega_N)^T, \\ \mathcal{A}(\sum_k n_k P_k) &= \sum_k n_k \mathcal{A}(P_k), \quad P, P_k \in \mathcal{K}_N,\end{aligned}$$

the Riemann theta function is defined as [3, 15, 16]

$$\begin{aligned}\theta(P, D_{\underline{u}(x, t_m)}) &= \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(1)}), \\ \theta(P, D_{\underline{v}(x, t_m)}) &= \theta(\underline{\Lambda} - \mathcal{A}(P) + \underline{\rho}^{(2)}),\end{aligned}$$

where  $\underline{\Lambda} = (\Lambda_1, \dots, \Lambda_N)$  is defined by

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{i=1, i \neq j}^N \int_{a_i} \omega_i \int_{Q_0}^P \omega_j, \quad j = 1, \dots, N.$$

In order to derive the algebro-geometric solution of coupled Burgers equation(1.1), now we turn to the asymptotic properties of the meromorphic function  $\phi$  and Baker-Akhiezer function  $\psi_1$ .

**Lemma 4.1.** Suppose that  $u(x, t_m), v(x, t_m) \in C^\infty(\mathbb{R}^2)$  satisfy the coupled Burgers equation (1.1). Moreover, let  $P = (\lambda, y) \in \mathcal{K}_N \setminus \{P_{\infty \pm}\}$ ,  $(x, x_0) \in \mathbb{R}^2$ . Then

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{u-v}{2}\zeta + [\frac{(u-v)x}{4} + \frac{v(u-v)}{2}]\zeta^2 + O(\zeta^3) & \text{as } P \rightarrow P_{\infty+}, \\ \frac{-2}{u+v}\zeta^{-1} + \frac{2v}{u+v} - \frac{(u+v)x}{(u+v)^2} + O(\zeta) & \text{as } P \rightarrow P_{\infty-}, \end{cases} \quad (4.18)$$

and

$$\psi_1(P, x, x_0, t_m, t_{m,0}) \underset{\zeta \rightarrow 0}{=} \begin{cases} \exp(-\zeta^{-1}(x - x_0) - 2\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty+}, \\ \exp(\zeta^{-1}(x - x_0) + 2\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty-}. \end{cases} \quad (4.19)$$

**Proof.** We first prove  $\phi$  satisfies the Riccati-type equations

$$\phi_x(P) + 2(-\lambda + v)\phi(P) - (u + v)\phi^2(P) + (u - v) = 0. \quad (4.20)$$

The local coordinates  $\zeta = \lambda^{-1}$  near  $P_{\infty\pm}$ , from (4.4), (2.11), we have

$$\begin{aligned}\phi_x &= \frac{G_x F - (y+G)F_x}{F^2} \\ &= \frac{(u+v)H - (u-v)F}{F} - \phi \frac{2F(-\lambda+v) - 2(u+v)G}{F} \\ &= -(u-v) - 2(-\lambda+v)\phi + \frac{(u+v)H + 2(u+v)G\phi}{F},\end{aligned}\tag{4.21}$$

$$\phi^2 = \frac{y^2 + 2yG + G^2}{F^2} = \frac{2G^2 + FH + 2yG}{F^2} = \frac{2G\phi + H}{F},\tag{4.22}$$

according to (4.21) and (4.22), we have (4.20). And then, inserting the ansatz  $\phi \underset{\lambda \rightarrow 0}{=} \phi_1 \lambda^{-1} + \phi_2 \lambda^{-2} + O(\lambda^{-3})$  into (4.20), we get the first line of (4.18). Inserting the ansatz  $\phi \underset{\lambda \rightarrow 0}{=} \phi_{-1} \lambda + \phi_0 + \phi_1 \lambda^{-1} + O(\lambda^{-2})$  into (4.20), we get the second line of (4.18). In the following, we will prove (4.19). From (4.7) and (4.18)

$$\begin{aligned}& \exp(\int_{x_0}^x ((-\lambda - v(x', t_m)) - (u(x', t_m) + v(x', t_m))\phi) dx') \\ &= \exp(\int_{x_0}^x ((-\zeta^{-1} - v) - (u+v)\phi) dx') \\ &\underset{\zeta \rightarrow 0}{=} \begin{cases} \exp(\int_{x_0}^x (-\zeta^{-1} - v) - (u+v)(\frac{u-v}{2}\zeta + [\frac{(u-v)_x}{4} + \frac{v(u-v)}{2}]\zeta^2 + O(\zeta^3)) \, as \, P \rightarrow P_{\infty+}, \\ \exp(\int_{x_0}^x (-\zeta^{-1} - v) - (u+v)(\frac{-2}{u+v}\zeta^{-1} + \frac{2v}{u+v} - \frac{(u+v)_x}{(u+v)^2} + O(\zeta)) \, as \, P \rightarrow P_{\infty-}, \end{cases} \\ &\underset{\zeta \rightarrow 0}{=} \begin{cases} \exp(-\zeta^{-1}(x - x_0) + O(1)) \, as \, P \rightarrow P_{\infty+}, \\ (\frac{u(x)+v(x)}{u(x_0)+v(x_0)} + O(\zeta)) \exp(\zeta^{-1}(x - x_0) + O(1)) \, as \, P \rightarrow P_{\infty-}. \end{cases}\end{aligned}\tag{4.23}$$

From (4.1) and (3.9), we have

$$\begin{aligned}y &= \mp \sqrt{R(\lambda)} \\ &= \mp 2 \prod_{j=1}^{2N+2} (\lambda - \lambda_j)^{\frac{1}{2}} \\ &= \mp 2 \zeta^{-N-1} \prod_{j=1}^{2N+2} (1 - \lambda_j \zeta)^{\frac{1}{2}} \\ &\underset{\zeta \rightarrow 0}{=} \mp 2 \zeta^{-N-1} \prod_{j=1}^{2N+2} (1 + \epsilon_1 \zeta + \epsilon_2 \zeta^2 + O(\zeta^3)) \, as \, P \rightarrow P_{\infty\pm},\end{aligned}\tag{4.24}$$

where  $\epsilon_1 = -\frac{1}{2} \sum_{j=1}^{2N+2} \lambda_j$ ,  $\epsilon_2 = \frac{1}{2} \sum_{j < k} \lambda_j \lambda_k - \frac{1}{8} (\sum_{j=1}^{2N+2} \lambda_j)^2$ . From (3.1) and (3.4), we can derive

$$\begin{aligned}F^{-1} &= \frac{1}{2(u+v)} \prod_{j=1}^N \frac{1}{\lambda - u_j} \\ &= \frac{1}{2(u+v)} \zeta^N \prod_{j=1}^N \frac{1}{1 - u_j \zeta} \\ &\underset{\zeta \rightarrow 0}{=} \frac{1}{2(u+v)} \zeta^N (1 + \sum_{j=1}^N u_j \zeta + O(\zeta^2)) \, as \, P \rightarrow P_{\infty\pm},\end{aligned}\tag{4.25}$$

combining (4.18), (4.26), (4.27) we have

$$\begin{aligned}
& \exp(\int_{t_{m,0}}^{t_m} (V_{11}^{(m)}(\lambda, x_0, s) - V_{12}^{(m)}(\lambda, x_0, s)) \phi(P, x_0, s) ds) \\
&= \exp(\int_{t_{m,0}}^{t_m} (V_{11}^{(m)} - V_{12}^{(m)} \frac{y+G}{F}) ds) \\
&= \exp(\int_{t_{m,0}}^{t_m} (\frac{-y}{F} V_{12}^{(m)} + \frac{F t_m}{2F}) ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-N-1} (1 + O(\zeta)) \frac{\sum_{j=0}^m [(G_{j-1,x}^{(2)} - G_{j-1,x}^{(1)}) - 2(u+v)G_{j-1}^{(2)}] \lambda^{m-j}}{\sum_{j=0}^N f_j \lambda^{N-j}} + \frac{u_{t_m} + v_{t_m}}{2(u+v)} ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-N-1} (1 + O(\zeta)) \frac{\sum_{j=0}^m [(G_{j-1,x}^{(2)} - G_{j-1,x}^{(1)}) - 2(u+v)G_{j-1}^{(2)}] \zeta^{j-m}}{\sum_{j=0}^N f_j \zeta^{j-N}} + \frac{u_{t_m} + v_{t_m}}{2(u+v)} ds) \quad (4.26) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-m-1} (1 + O(\zeta)) \frac{\sum_{j=0}^m [(G_{j-1,x}^{(2)} - G_{j-1,x}^{(1)}) - 2(u+v)G_{j-1}^{(2)}] \zeta^j}{\sum_{j=0}^N f_j \zeta^j} + \frac{u_{t_m} + v_{t_m}}{2(u+v)} ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \exp(\int_{t_{m,0}}^{t_m} (\pm \zeta^{-m-1} (1 + O(\zeta)) \frac{(G_{-1,x}^{(2)} - G_{-1,x}^{(1)}) - 2(u+v)G_{-1}^{(2)}}{f_0} + \frac{u_{t_m} + v_{t_m}}{2(u+v)} ds) \\
&\stackrel{\zeta \rightarrow 0}{=} \begin{cases} \exp(-\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty+}, \\ \exp(\zeta^{-m-1}(t_m - t_{m,0}) + O(1)) & \text{as } P \rightarrow P_{\infty-}, \end{cases}
\end{aligned}$$

according to the definition of  $\psi_1$  in (4.7), (4.25) and (4.26), we can obtain (4.19).  $\square$

Next, we shall derive the representation of  $\phi, \psi_1, \psi_2, u(x, t_m), v(x, t_m)$  in term of the Riemann theta function. Let  $\omega_{P_{\infty+}, P_{\infty-}}^{(3)}$  be the normalized differential of the third kind holomorphic on  $\mathcal{K}_N \setminus \{P_{\infty+}, P_{\infty-}\}$  with simples at  $P_{\infty+}$  and  $P_{\infty-}$  and residues 1 and  $-1$  respectively,

$$\omega_{P_{\infty+}, P_{\infty-}}^{(3)} = \frac{1}{y} \prod_{j=1}^N (\lambda - \lambda_j) d\lambda \stackrel{\zeta \rightarrow 0}{=} (\pm \zeta^{-1} + O(1)) d\zeta \quad \text{as } P \rightarrow P_{\infty\pm}, \quad (4.27)$$

here the constants  $\{\lambda_j | \lambda_j \in \mathbb{C}, j = 1, \dots, N\}$  are uniquely determined by the normalization

$$\int_{a_j} \omega_{P_{\infty+}, P_{\infty-}}^{(3)} = 0, \quad j = 1, \dots, N, \quad (4.28)$$

and  $\zeta$  in (4.30) denotes the local coordinate  $\zeta = \lambda^{-1}$  for  $P$  near  $P_{\infty\pm}$ . Moreover,

$$\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \stackrel{\zeta \rightarrow 0}{=} \ln(\zeta) - \ln \omega_0 + O(\zeta) \quad \text{as } P \rightarrow P_{\infty+}, \quad (4.29)$$

and

$$\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)} \stackrel{\zeta \rightarrow 0}{=} -(\ln(\zeta) - \ln \omega_0 + O(\zeta)) \quad \text{as } P \rightarrow P_{\infty-}. \quad (4.30)$$

Let  $\omega_{P_{\infty\pm}, r}^{(2)}$ ,  $r \in \mathbb{N}_0$ , be normalized differentials of the second kind with a unique pole at  $P_{\infty\pm}$ , and principal part is  $\zeta^{-2-r} d\zeta$  near  $P_{\infty\pm}$ , satisfying

$$\int_{a_j} \omega_{P_{\infty\pm}, r}^{(2)} = 0, \quad j = 1, \dots, N,$$

then we can define  $\Omega_0^{(2)}$  and  $\Omega_{m-1}^{(2)}$  by

$$\Omega_0^{(2)} = \omega_{P_{\infty-},0}^{(2)} - \omega_{P_{\infty+},0}^{(2)}, \quad (4.31)$$

$$\Omega_{m-1}^{(2)} = \sum_{l=0}^m \alpha_m(l+1)(\omega_{P_{\infty-},l}^{(2)} - \omega_{P_{\infty+},l}^{(2)}), \quad (4.32)$$

where  $\alpha_{m-1-l}$ ,  $j = 0, \dots, m-1$  are the integral constants in (2.15), so we have

$$\int_{a_j}^P \Omega_0^{(2)} = 0, \int_{a_j}^P \Omega_{m-1}^{(2)}, \quad j = 1, \dots, N, \quad (4.33)$$

$$\int_{Q_0}^P \Omega_0^{(2)} \underset{\zeta \rightarrow 0}{=} \mp(\zeta^{-1} + e_{0,0} + O(\zeta)) \quad \text{as } P \rightarrow P_{\infty\pm}, \quad (4.34)$$

$$\int_{Q_0}^P \Omega_{m-1}^{(2)} \underset{\zeta \rightarrow 0}{=} \mp\left(\sum_{l=0}^m \alpha_{m-1}\zeta^{-1-l} + e_{m,0} + O(\zeta)\right) \quad \text{as } P \rightarrow P_{\infty\pm}, \quad (4.35)$$

for some constants  $e_{0,0}, e_{m,0} \in \mathbb{C}$ .

If  $D_{\underline{u}(x,t_m)}$  or  $D_{\underline{v}(x,t_m)}$  in (4.5) is assumed to be nonspecial[3], then according to Riemann's theorem[3, 16], the definition and asymptotic properties of the meromorphic function  $\phi(P, x, t_m)$ ,  $\phi(P, x, t_m)$  has expressions of the following type

$$\phi(P, x, t_m) = N(x, t_m) \frac{\theta(P, D_{\underline{v}(x,t_m)})}{\theta(P, D_{\underline{u}(x,t_m)})} \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right), \quad (4.36)$$

where  $N(x, t_m)$  is independent of  $P \in \mathcal{K}_N$ .

**Theorem 4.1.** Let  $P = (\lambda, y) \in \mathcal{K}_N \setminus P_{\infty\pm}$ ,  $(x, t_m), (x_0, t_{m,0}) \in \Omega$ , where  $\Omega \subseteq \mathbb{R}^2$  is open and connected. Suppose  $u(\cdot, t_m), v(\cdot, t_m) \in C^\infty(\Omega)$ ,  $u(x, \cdot), v(x, \cdot) \in C^1(\Omega)$ ,  $x \in \mathbb{R}, t_m \in \mathbb{R}$ , satisfy the equation (1.1), and assume that  $\lambda_j, 1 \leq j \leq 2N+2$  in (3.9) satisfy  $\lambda_j \in \mathbb{C}$  and  $\lambda_j \neq \lambda_k$  for  $j \neq k$ . Moreover, suppose that  $D_{\underline{u}}$  or equivalently,  $D_{\underline{v}}$ , is nonspecial for  $(x, t_m) \in \Omega$ . Then

$$\phi(P, x, t_m) = N(x, t_m) \frac{\theta(P, D_{\underline{v}(x,t_m)})}{\theta(P, D_{\underline{u}(x,t_m)})} \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right), \quad (4.37)$$

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) &= \frac{\theta(P_{\infty+}, D_{\underline{u}(x_0, t_{m,0})})\theta(P, D_{\underline{u}(x, t_m)})}{\theta(P_{\infty+}, D_{\underline{u}(x, t_m)})\theta(P, D_{\underline{u}(x_0, t_{m,0})})} \\ &\times \exp\left(\left(\int_{Q_0}^P \Omega_0^{(2)} + e_{0,0}\right)(x - x_0) + \left(\int_{Q_0}^P \Omega_m^{(2)} + e_{m,0}\right)(t_m - t_{m,0})\right), \end{aligned} \quad (4.38)$$

$$\begin{aligned} \psi_2(P, x, x_0, t_m, t_{m,0}) &= N(x, t_m) \frac{\theta(P, D_{\underline{v}(x,t_m)})\theta(P_{\infty+}, D_{\underline{u}(x_0, t_{m,0})})\theta(P, D_{\underline{u}(x, t_m)})}{\theta(P, D_{\underline{u}(x, t_m)})\theta(P_{\infty+}, D_{\underline{u}(x, t_m)})\theta(P, D_{\underline{u}(x_0, t_{m,0})})} \\ &\times \exp\left(\left(\int_{Q_0}^P \Omega_0^{(2)} + e_{0,0}\right)(x - x_0) + \left(\int_{Q_0}^P \Omega_m^{(2)} + e_{m,0}\right)(t_m - t_{m,0})\right) \\ &\times \exp\left(\int_{Q_0}^P \omega_{P_{\infty+}, P_{\infty-}}^{(3)}\right). \end{aligned} \quad (4.39)$$

Finally,  $u(x, t_m)$  is of the form

$$u(x, t_m) = \omega_0 N(x, t_m) \frac{\theta(P_{\infty+}, D_{\underline{v}(x,t_m)})}{\theta(P_{\infty+}, D_{\underline{u}(x,t_m)})} - \frac{1}{\omega_0 N(x, t_m)} \frac{\theta(P_{\infty-}, D_{\underline{u}(x,t_m)})}{\theta(P_{\infty-}, D_{\underline{v}(x,t_m)})}, \quad (4.40)$$

and  $v(x, t_m)$  is of the form

$$v(x, t_m) = -\frac{1}{\omega_0 N(x, t_m)} \frac{\theta(P_{\infty-}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty-}, D_{\hat{v}}(x, t_m))} - \omega_0 N(x, t_m) \frac{\theta(P_{\infty+}, D_{\hat{v}}(x, t_m))}{\theta(P_{\infty+}, D_{\hat{u}}(x, t_m))}, \quad (4.41)$$

and  $N(x, t_m)$  is determined by

$$\begin{aligned} & \omega_0 N(x, t_m) \frac{\theta(P_{\infty+}, D_{\hat{v}}(x, t_m))}{\theta(P_{\infty+}, D_{\hat{u}}(x, t_m))} + \frac{1}{\omega_0 N(x, t_m)} \frac{\theta(P_{\infty-}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty-}, D_{\hat{v}}(x, t_m))} \\ &= \partial_x \ln \frac{\theta(P_{\infty+}, D_{\hat{u}}(x_0, t_{m,0})) \theta(P_{\infty-}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty-}, D_{\hat{u}}(x_0, t_{m,0})) \theta(P_{\infty+}, D_{\hat{u}}(x, t_m))} - 2e_{0,0}. \end{aligned} \quad (4.42)$$

**proof.** We start with the proof of the theta function representation (4.38). Without loss of generality, it suffices to treat the special case of (2.15) when  $\alpha_0 = 2, \alpha_k = 0, 1 \leq k \leq N$ . First, we assume

$$u_j(x, t_m) \neq u_k(x, t_m), \text{ for } j \neq k, \text{ and } (x, t_m) \in \tilde{\Omega} \quad (4.43)$$

for appropriate  $\tilde{\Omega} \subseteq \Omega$ , and define the right-hand side of (4.38) to be  $\tilde{\psi}_1$ . In order to prove  $\psi = \tilde{\psi}$ , we investigate the local zeros and poles of  $\psi_1$ . From (3.4), (3.15), (3.16), (3.17), (3.18), (4.4), we have

$$\begin{aligned} (u(x', t_m) + v(x', t_m)) \phi(P, x', t_m) &= \frac{y(\hat{u}_j(x', t_m))}{\prod_{k=1, k \neq j} (u_j(x', t_m) - u_k(x', t_m))} \frac{1}{\lambda - u_j(x', t_m)} \\ &= \frac{y_{j, x'}}{y - u_j(x', t_m)} \\ &= -\partial_{x'} \ln(y - u_j(x', t_m)) + O(1). \end{aligned} \quad (4.44)$$

And similarly

$$V_{12}^{(m)}(\lambda, x_0, s) \phi(P, x_0, s) \underset{P \rightarrow \hat{u}_j(x_0, s)}{=} -\partial_s \ln(y - u_j(x_0, s)) + O(1), \quad (4.45)$$

then (4.44) and (4.45) together with (4.7) yields

$$\psi_1(P, x, x_0, t_m, t_{m,0}) = \begin{cases} (\lambda - u_j(x, t_m)) O(1), & \text{as } P \rightarrow \hat{u}_j(x, t_m) \neq \hat{u}_j(x_0, t_{m,0}) \\ O(1), & \text{as } P \rightarrow \hat{u}_j(x, t_m) = \hat{u}_j(x_0, t_{m,0}) \\ (\lambda - u_j(x_0, t_{m,0}))^{-1} O(1), & \text{as } P \rightarrow \hat{u}_j(x_0, t_{m,0}) \neq \hat{u}_j(x, t_m) \end{cases} \quad (4.46)$$

where  $P = (\lambda, t_m) \in \mathcal{K}_N, (x, t_m), (x_0, t_{m,0}) \in \tilde{\Omega}$  and  $O(1) \neq 0$ . Hence  $\psi_1$  and  $\tilde{\psi}_1$  have identical zeros and poles on  $\mathcal{K}_N \setminus \{P_{\infty\pm}\}$ , which are all simple by hypothesis (4.43). It remains to study the behavior of  $\psi_1$  and  $\tilde{\psi}_1$  near  $P_{\infty\pm}$ , by (4.19), (4.34), (4.35), (4.38), we can easily find that  $\psi_1$  and  $\tilde{\psi}_1$  share the same singularities and zeros, and the Riemann-Roch-type uniqueness[3] proves that  $\psi_1 = \tilde{\psi}_1$ , hence (4.38) holds subject to (4.43).

Substituting (4.29), (4.30) into (4.36) and comparing with (4.18), we obtain

$$u(x, t_m) - v(x, t_m) = 2\omega_0 N(x, t_m) \frac{\theta(P_{\infty+}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty+}, D_{\hat{v}}(x, t_m))}, \quad (4.47)$$

$$u(x, t_m) + v(x, t_m) = \frac{-2}{\omega_0 N(x, t_m)} \frac{\theta(P_{\infty-}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty-}, D_{\hat{v}}(x, t_m))}, \quad (4.48)$$

according to (4.48), we have (4.40) and (4.41), and  $\psi_2$  in (4.39) from  $\psi_2 = -\phi\psi_1$ . Reexamining the asymptotic behavior of  $\psi_1$  near  $P_{\infty-}$  yields

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) = & \exp\left(\int_{x_0}^x (-v(x', t_m) dx' + O(\zeta))\right) \\ & \times \exp(\zeta^{-1}(x - x_0) + \zeta^{-m-1}(t_m - t_{m,0}) + O(1)). \end{aligned} \quad (4.49)$$

On the other hand, according to (4.38), (4.34), (4.35), we have

$$\begin{aligned} \psi_1(P, x, x_0, t_m, t_{m,0}) = & \frac{\theta(P_{\infty+}, D_{\hat{u}}(x_0, t_{m,0}))\theta(P_{\infty-}, D_{\hat{u}}(x, t_m))}{\theta(P_{\infty+}, D_{\hat{u}}(x, t_m))\theta(P_{\infty-}, D_{\hat{u}}(x_0, t_{m,0}))} \\ & \times \exp((\zeta^{-1} + 2e_{0,0} + O(\zeta))(x - x_0) + (\zeta^{-m-1} + 2e_{m,0} + O(\zeta))(t_m - t_{m,0})), \end{aligned} \quad (4.50)$$

a comparison of (4.49) and (4.50) proves (4.42).

Hence, we prove this theorem on  $\tilde{\Omega}$ . The extension of all these results from  $\tilde{\Omega}$  to  $\Omega$  follows by continuity of the Abel map and the nonspecial nature of  $D_{\hat{u}}$  or  $D_{\hat{v}}$  on  $\Omega$ .  $\square$

Therefore, the algebro-geometric solution of (1.1) is (4.40) and (4.41) for  $m = 1$ .

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## References

- [1] S. P. Novikov, S. V. Manakov, L. P. Pitaeskii and V. E. Zakharov, *Theory of solitons, the inverse scattering methods*, Nauka, Moscow, 1980.
- [2] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.
- [3] F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions(Volnme I)*, Cambridge University Press, Cambridge, 2003.
- [4] F. Gesztesy, H. Holden, J. Michor and G. Teschl, *Soliton Equations and their Algebro-Geometric Solutions(Volnme II)*, Cambridge University Press, Cambridge, 2008.

- [5] S. P. Novikov, A periodic problem for the KdV equation. *Functional analysis and its applications*, 1974**8**: 236-246.
- [6] B. A. Dubrovin, Periodic problem for the Kortweg-de Vries equation in the class of finite-band potentials, *Functional analysis and its applications*, 1975**9**: 265-273.
- [7] P. D. Lax, Periodic solutions of the KdV equation, *Communications on Pure and Applied Mathematics*, 1975**28**: 141-188.
- [8] H. P. McKean and P. Moerbeke, The spectrum of Hill's equation, *Inventiones mathematicae*, 1975**30**: 217-274.
- [9] X. G. Geng, H. H. Dai and C. W. Cao, Algebro-geometric constructions of the discrete Ablowitz-Ladik flows and applications, *Journal of Mathematical Physics*, 2003**44**: 4573-4588.
- [10] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, Society for Industrial Mathematics, Philadelphia, 1981.
- [11] R. Hirota, *The direct methods in soliton theory*, Cambridge University Press, Cambridge, England, 2004.
- [12] C. W. Cao and X. G. Geng, in *Nonlinear physics, Research Reports in physics*, edited by C. H. Gu *et al*, Springer, Berlin, 1990, 68-78.
- [13] Y. Hou, E. G. Fan and P. Zhao, Algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy, *Journal of Mathematical Physics*, 2013**54**: 073505.
- [14] B. Xue, X. G. Geng and F. Li, Quasiperiodic solutions of Jaulent-Miodek equations with a negative flow, *Journal of Mathematical Physics*, 2012**53**: 1-16.
- [15] C. L. Siegel, *Topics in Complex Function Theory*. **2nd**, Wiley, New York, 1971.
- [16] P. Griffiths and J. Harris, *Principles of Algebraic geometry*, Wiley, New York, 1994.